Problem 1. .

(a) Let $U_1 = T_1$, $U_2 = T_2 - T_1$, $U_3 = T_3 - T_2$. Since $U \sim \exp(2)$,

$$P(T_1 + T_2 < T_3) = P(2U_1 + U_2 < U_1 + U_2 + U_3) = P(U_1 < U_3)$$

$$= \int_0^\infty \int_0^t 2e^{-2u}\, du\, 2e^{-2t} dt = \int_0^\infty \left(-e^{-2u} \Big|_0^t \right) 2e^{-2t} dt$$

$$= 2\int_0^\infty \left(e^{-2t} - e^{-4t}\right) dt = \left(-e^{-2t} + \tfrac{1}{2}e^{-4t}\right)\Big|_0^\infty = \tfrac{1}{2}$$

(b) $E N^2(S) = E\left[E N^2(S) \mid S\right] = E\left[2S + 4S^2\right] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} = \frac{7}{3}$

(c) $P\left(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\right) = P\left(N((n+1)^2) - N(n^2)\right)$

$$= P\left(X_{n+1} = j \mid X_n = i\right) = \frac{2(2n+1)^{j-i}}{(j-i)!} e^{-2(2n+1)} \mathbb{1}_{(j \geq i \geq i_{n-1} \geq \dots \geq i_0)}$$

The RVs $(X_n)_n$ form a first-order homogeneous Markov chain.

Problem 2.

(a) By the Cesàro means theorem, for any $\omega \in \Omega$ such that $\lim_{n\to\infty} X_n = 0$, also $\lim_{n\to\infty} \frac{1}{n} S_n = 0$; thus $P(\frac{1}{n} S_n \to 0) = 1$.

(b) Consider a sequence $X_n(\omega) \to 0$; suppose that $|X_n(\omega)| < \varepsilon$ for all $n \geq N = N(\omega)$. Then, using the estimate of partial sums of the harmonic series,

$$\left| \frac{1}{\log n} \sum_{k=1}^n \frac{X_k(\omega)}{k} \right| \leq \varepsilon \left| \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \right| \leq \varepsilon \frac{\log n + 1}{\log n} \leq 2\varepsilon$$

and thus, $\left| \frac{1}{\log n} \sum_{k=1}^n \frac{X_k(\omega)}{k} \right| \to 0$.

(c) As above in part (b)

$$\sum_{K=2}^{n} \frac{1}{K \log K} \leq \int_{3}^{n+1} \frac{dx}{x \log x} = \int_{\log 3}^{\log(n+1)} \frac{du}{u} = \log\log(n+1)$$

The remainder of the proof is the same as in Part (b).

Note: In the problem statement the sum starts at $k=1$, but for $k=1$ the summation term is not well defined. Take instead $k=2$.

(d) To show this take $X_n \sim$ Bernoulli

$$P(X_n = 0) = 1 - \frac{1}{n} \;;\; P(X_n = 2^n) = \frac{1}{n}, \quad n \geq 1$$

Then $X_n \xrightarrow{P} 0$, but

$$P\left(\frac{1}{n}\sum_{K=1}^{n} X_K < \varepsilon\right) \geq P\left(X_{\lceil \frac{n}{2} \rceil} = X_{\lceil \frac{n}{2} \rceil + 1} = \ldots X_n = 0\right) \geq \left(1 - \frac{1}{n}\right)^{n/2} \not\to 0$$

(e) Assuming that $Var(X) < \infty$, we use SLLN to claim that

$$\frac{X_1 + \ldots + X_n}{n} \xrightarrow{a.s.} EX \;;\; \frac{X_1^2 + \ldots + X_n^2}{n} \xrightarrow{a.s.} EX^2$$

Since the intersection of two events of prob. 1 has probability 1,

$$\frac{X_1 + \ldots + X_n}{X_1^2 + \ldots + X_n^2} \xrightarrow{a.s.} \frac{EX}{EX^2}$$

Problem 3.

(a) $E|X_n| = EX_n = (EY)^n = 1.$

$$E(X_{n+1} | X_1^n) = E(X_n Y_{n+1} | X_1^n) = X_n$$

which proves that $(X_n)_n$ form a martingale with respect to the natural filtration.

(b) $(X_n, \mathcal{F}_n)_n$ forms an L_1-bounded martingale; thus there is an RV X_∞ on $(\Omega, \mathcal{F}_\infty)$ s.t. $X_n \xrightarrow{a.s.} X_\infty$.

To find the limit, consider $\log X_n = \sum_{k=1}^{n} \log Y_k$

We have $\dfrac{\log X_n}{n} = \dfrac{\sum \log Y_k}{n} \xrightarrow{a.s.} E \log Y_k = \frac{1}{2}\left(\log \frac{1}{4} \cdot \frac{3}{4}\right) = \frac{1}{2} \log \frac{3}{16} < 0.$

Thus $P\left(\lim_{n\to\infty} \dfrac{\log X_n}{n} = \log \frac{\sqrt{3}}{4}\right) = 1$

or $\log X_n \xrightarrow{a.s.} -\infty$, or $X_n \xrightarrow{a.s.} X_\infty = 0$

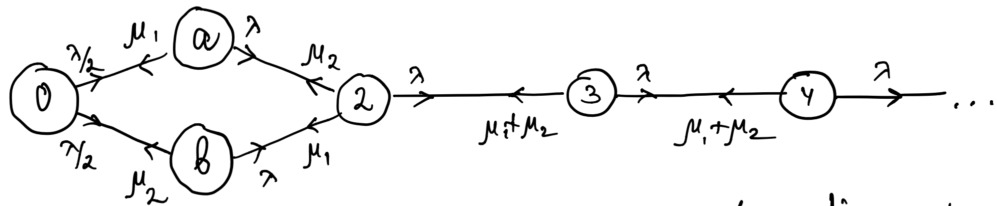
At the same time, $X_n \not\xrightarrow{L_1} X_\infty$, i.e. $E|X_n - 0| \not\to 0$

so the sequence does not converge in L_1.

A necessary condition for L_1 convergence is $E(X_\infty | X_1^n) = X_n$, but it is clearly not satisfied.

Problem 4.

(a) The Markov chain can be graphically represented as follows:



where on the edges we write exponential transition rates.

The generator matrix Q has the form

$$Q = \begin{array}{c} \\ 0 \\ a \\ b \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & a & b & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -\lambda & \lambda/2 & \lambda/2 & 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -\mu_1-\lambda & 0 & \lambda & 0 & 0 & 0 & \cdots \\ \mu_2 & 0 & -\mu_2-\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & \mu_2 & \mu_1 & -\lambda-\mu_1-\mu_2 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu_1+\mu_2 & -\lambda-\mu_1-\mu_2 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu_1+\mu_2 & -\lambda-\mu_1-\mu_2 & \lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{bmatrix} \end{array}$$

where q_{ij} represents the exponential rate of moving $i \to j$.

Write $\pi Q = 0$ in detail. The 3 first relations are:

$$(1) \quad \begin{cases} \pi_0 \lambda = \mu_1 \pi_a + \mu_2 \pi_b \\ \pi_0 \frac{\lambda}{2} - (\mu_1+\lambda)\pi_a + \mu_2 \pi_2 = 0 \\ \pi_0 \frac{\lambda}{2} - (\mu_2+\lambda)\pi_b + \mu_1 \pi_2 = 0 \end{cases}$$

Eliminating π_b and π_2, we obtain

$$0 = \underline{\lambda\pi_0} - \lambda\pi_a - \underline{\mu_1\pi_a} + \mu_2\pi_2 - \underline{\mu_2\pi_b} - \lambda\pi_b + \mu_1\pi_2$$

$$= -\lambda(\pi_a + \pi_b) + (\mu_1+\mu_2)\pi_2$$

i.e., $$\pi_2 = \frac{\lambda}{\mu_1+\mu_2}(\pi_a+\pi_b) = \frac{\lambda}{\mu_1+\mu_2}\left(\pi_a + \frac{\lambda\pi_0 - \mu_1\pi_a}{\mu_2}\right)$$

Substitute this into the second equation in (1). Simplifying we obtain

(2) $\pi_0 \frac{\lambda}{2} = \pi_a\mu_1$, i.e., $\boxed{\pi_0 Q_{0a} = \pi_a Q_{a0}}$

Now substitute $\pi_a = \pi_0 \frac{\lambda}{2\mu_1}$ into the first equation in (1):

(3) $\pi_0\lambda = \pi_0\frac{\lambda}{2} + \mu_2\pi_b$, i.e., $\boxed{\pi_0 Q_{0b} = \pi_b Q_{b0}}$

Eliminating π_0 from the second and third equations in (1), we obtain

(4) $$\boxed{\begin{aligned} \pi_a Q_{a2} &= \pi_2 Q_{2a} \\ \pi_b Q_{b2} &= \pi_2 Q_{2b} \end{aligned}}$$

Finally let us states $i \geq 2$. The 4th relation gives

$$\pi_a\lambda + \pi_b\lambda - \pi_2(\lambda+\mu_1+\mu_2) + \pi_3(\mu_1+\mu_2) = 0;$$

substituting π_a and π_b from (2)-(3), we obtain

$$\pi_2\lambda = \pi_3(\mu_1+\mu_2)$$

and generally

(5) $\boxed{\pi_i Q_{i,i+1} = \pi_{i+1} Q_{i+1,i}, \quad i \geq 2}$

This shows that the chain is reversible.

(b) The chain is irreducible, so its stationary distribution π is at the same time the limiting distribution.

The stationary distribution is easily computed from Eqns. (2)-(5) in part (a). Specifically,

$$(6)\begin{cases} \pi_a = \frac{\lambda}{2\mu_1}\pi_0 \ ; \ \pi_b = \frac{\lambda}{2\mu_2}\pi_0 \\ \pi_2 = \frac{\lambda^2}{2(\mu_1+\mu_2)}\pi_0 \\ \frac{\pi_i}{\pi_{i+1}} = \frac{\lambda}{\mu_1+\mu_2}, \quad i \geq 2 \end{cases}$$

Denote $x := \frac{\lambda}{\mu_1+\mu_2}$ and assume that $x<1$ (if not, then the queue increases without limit, and there is no stationarity).

Normalization condition gives

$$\pi_0\left(1+\frac{\lambda}{2\mu_1}+\frac{\lambda}{2\mu_2}+\frac{\lambda^2}{2(\mu_1+\mu_2)}\right)+\pi_2\sum_{i\geq 1}x^i$$

$$= \pi_0\left(1+\frac{\lambda}{2\mu_1}+\frac{\lambda}{2\mu_2}+\frac{\lambda^2}{2(\mu_1+\mu_2)}\ \frac{1}{1-x}\right) = \pi_0\left(1+\frac{\lambda}{2\mu_1}+\frac{\lambda}{2\mu_2}+\frac{\lambda^2}{2(\mu_1+\mu_2-\lambda)}\right)$$

$$= 1$$

This yields the value of π_0, and the remaining components of the distribution are found from (6) above.

Problem 5.

(a) Let us compute $EX(t)$:

$$EX(t) = EY^3 t = t\int_{-1}^{1} y^3 \frac{dy}{2} = 0.$$

The auto-covariance function is found as

$$E\left(X(t)\,X(t+s)\right) = t(t+s)\,EY^6 = \frac{1}{7}t(t+s).$$

This depends on t, so $X(t)$ is not even WSS, let alone stationary.

(b) Take t_1, t_2, t_3, t_4 to satisfy $0<t_1<t_2<t_3<t_4$, and let

$$Z_1 = X(t_2)-X(t_1)$$

$$Z_2 = X(t_4)-X(t_3)$$

$$EZ_1 = Ee^Y t_2 - Ee^Y t_1 = (t_2-t_1)\int_0^1 e^y\,dy = (t_2-t_1)(e-1).$$

$$EZ_2 = (t_4-t_3)(e-1)$$

$$E(Z_1Z_2) = E\left((t_2-t_1)(t_4-t_3)e^{2Y}\right) = (t_2-t_1)(t_4-t_3)\frac{e^2-1}{2}$$

$$\neq EZ_1\,EZ_2$$

and thus the increments are correlated, so not independent.

Now consider the distribution of $W(t_1,t_2,s) := X(t_2+s)-X(t_1+s)$:

$$F_W(x) = P(X(t_2+s)-X(t_1+s)<x) = P\left(e^Y(t_2-t_1)<x\right)$$

This probability does not depend on s, so the increments are stationary.